Carleton College, winter 2013
Math 232, Solutions to review problems and practice midterm 1
Prof. Jones

Solutions to review problems:

Chapter 1:

1. F	6. F	11. T	16. F	21. F	26. T	34. T	42. T
2. T	7. T	12. T	17. T	22. T	27. F	35. F	43. T
3. T	8. T	13. F	18. T	23. F	31. F	37. F	
4. F	9. T	14. F	19. T	24. F	32. F	39. T	
5. F	10. F	15. F	20. T	25. F	33. T	41. T	

Chapter 2:

1. T	6. F	12. T	21. T	26. T	34. F	41. T	49. F
2. T	8. T	13. F	22. T	29. F	35. T	43. T	
3. F	9. T	15. F	23. T	30. T	38. T	44. F	53. T
4. T	10. F	18. T	24. F	32. T	39. F	45. T	
5. F	11. F	19. T	25. F	33. F	40. T	46. T	54. F

Chapter 3:

2. F	5. T	12. T	18. T	21. T	26. T	37. F
3. T	11. T	16 . T	20. F	22. T	29. T	41. T

Solutions to practice exam:

1. Find all solutions of the following system of linear equations.

$$\begin{array}{rrrrl} x_1 & +2x_2 & & +x_4 & = 0 \\ -3x_1 & & +x_3 & & = 1 \\ -x_1 & +4x_2 & +x_3 & +2x_4 & = 1 \end{array}$$

To find the solutions, we need to put this system into reduced row-echelon form. Adding 3 times the first equation to the second equation and 1 times the first equation to the third equation gives

$$\begin{array}{rrrrl} x_1 & +2x_2 & & +x_4 & = 0 \\ & +6x_2 & +x_3 & +3x_4 & = 1 \\ & +6x_2 & +x_3 & +3x_4 & = 1 \end{array}$$

Adding -1 times the second equation to the third equation and -1/3 times the second equation to the first equation and then multiplying the second equation by 1/6 gives

$$\begin{array}{ccccccc} x_1 & & -(1/3)x_3 & & = & -1/3 \\ & +x_2 & +(1/6)x_3 & +(1/2)x_4 & = & 1/6 \\ & & & 0 & = & 0 \end{array}$$

This is now in rref. We see that the leading variables are x_1 and x_2, while the free variables are x_3 and x_4. Setting $x_3 = t_1$ and $x_4 = t_2$, we get that the set of solutions is

$$\left\{ \left(\frac{1}{3}t_1 - \frac{1}{3}, -\frac{1}{6}t_1 - \frac{1}{2}t_2 + \frac{1}{6}, t_1, t_2 \right) \in \mathbb{R}^4 \mid t_1, t_2 \in \mathbb{R} \right\}$$

2. Let A be an $n \times n$ matrix of rank n, and suppose that $\mathbf{b}$ is a vector in $\mathbb{R}^n$. Must there be $\mathbf{v} \in \mathbb{R}^n$ with $A\mathbf{v} = \mathbf{b}$? Prove or give a counter-example.

   Yes. An $n \times n$ matrix of rank n is invertible, and thus $\mathbf{v} = A^{-1}\mathbf{b}$ is the solution we seek.

3. For each of the following subsets W of a vector space V, determine if W is a subspace of V. In each case either prove that W is a subspace or give a concrete reason why it is not a subspace.

   (a) $V = \mathbb{R}^4$, and $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 = x_3 - x_2, \text{ and } x_4 = 0\}$

   Observe first that $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 + x_2 - x_3 = 0, x_4 = 0\}$. So W is the set of vectors $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{0}$, with $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. So W is the kernel of A, and thus is a subspace by a theorem from class.

   (b) $V = \mathbb{R}^4$, and $W = \{(x_1, x_2, x_3, x_4) \,|\, x_1 = x_3 - x_2, \text{ and } x_4 = x_1 x_3\}$

   It seems unlikely to be a subspace since the second condition $x_4 = x_1 x_3$ involves multiplication of variables and so is not linear. Indeed, we can show W is not closed under addition: $(1, 2, 3, 3) \in W$ and $(-1, 2, 1, -1) \in W$ but their sum $(0, 4, 4, 2)$ is not in W since it fails the condition $x_4 = x_1 x_3$. You could also proceed by showing that W is not closed under scalar multiplication: $(1, 2, 3, 3) \in W$, but $2(1, 2, 3, 3) = (2, 4, 6, 6)$ is not in W.

4. (a) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Show that $A^2 = 0$ and that $A + I_2$ is invertible.

   This is a direct computation, first of A^2, and then of the determinant of $A + I_2$, which is 1. Since $A + I_2$ has non-zero determinant, it is invertible.

   (b) Suppose that A is an $n \times n$ matrix with $A^2 = 0$. Must $A + I_n$ always be invertible? Either explain why or give a counter-example.

   Yes. We have $(A + I_n)(A - I_n) = A^2 + I_n A - A I_n + I_n I_n = 0 + A - A + I_n = I_n$, where we have used several properties of I_n. Thus there is a matrix B such that $(A + I_n)B = I_n$ (we can take $B = A - I_n$). So by a theorem from class, $A + I_n$ is invertible.

5. Find all the vectors in the kernel of each of the following linear transformations, and justify your answers.

   (a) The shear $T(\mathbf{x}) = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \mathbf{x}$.

   The matrix has non-zero determinant, and so is invertible. Thus the only $\mathbf{x}$ with $T(\mathbf{x}) = \mathbf{0}$ is $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$. So the kernel of T is $\{\mathbf{0}\}$.

   (b) Reflection about a plane in $\mathbb{R}^3$.

   This transformation is again invertible (it is its own inverse, since $T(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$), so it has kernel $\{\mathbf{0}\}$.

   (c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by projection onto the line $y = x$.

   The kernel consists of the vectors that project to $\mathbf{0}$, which are precisely those that lie along the line perpendicular to $y = x$. That line is $y = -x$, also known as the span of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   (d) $T(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times m$ matrix of rank m.

   We seek all solutions to $A\mathbf{x} = \mathbf{0}$. So we put the augmented matrix $[A|\mathbf{0}]$ into rref, and we find there is a leading 1 in each column (since the rank of the matrix equals the number of columns). Thus there are no free variables, and it follows that $A\mathbf{x} = \mathbf{0}$ has a unique solution. Hence the kernel of T consists only of $\mathbf{0}$.

6. Given subspaces W_1 and W_2 of $\mathbb{R}^n$, set

$$W_1 + W_2 = \{\mathbf{x} \in V : \mathbf{x} = \mathbf{w_1} + \mathbf{w_2} \text{ for some } \mathbf{w_1} \in W_1 \text{ and } \mathbf{w_2} \in W_2\}.$$

   Prove that $W_1 + W_2$ is a subspace of V.

   Let $\mathbf{x}$ and $\mathbf{y}$ be in $W_1 + W_2$, meaning that $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$ for some $\mathbf{x_1} \in W_1$, $\mathbf{x_2} \in W_2$ and $\mathbf{y} = \mathbf{y_1} + \mathbf{y_2}$ for some $\mathbf{y_1} \in W_1$, $\mathbf{y_2} \in W_2$. Then

$$\mathbf{x} + \mathbf{y} = \mathbf{x_1} + \mathbf{x_2} + \mathbf{y_1} + \mathbf{y_2} = (\mathbf{x_1} + \mathbf{y_1}) + (\mathbf{x_2} + \mathbf{y_2}).$$

   Since W_1 is a subspace, $\mathbf{x_1} + \mathbf{y_1} \in W_1$, and since W_2 is a subspace, $\mathbf{x_2} + \mathbf{y_2} \in W_2$. We've thus written $\mathbf{x} + \mathbf{y}$ as an element of W_1 plus an element of W_2, showing that $\mathbf{x} + \mathbf{y} \in W_1 + W_2$.

   Now let $\mathbf{x} \in W$ and c be a scalar, so that $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$ for some $\mathbf{x_1} \in W_1$, $\mathbf{x_2} \in W_2$. Then $c\mathbf{x} = c(\mathbf{x_1} + \mathbf{x_2}) = c\mathbf{x_1} + c\mathbf{x_2}$, and this last expression is the sum of an element of W_1 and an element of W_2, and so is in $W_1 + W_2$.

7. Let $V = \mathbb{R}^2$ and $S = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$.

   (a) Show that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in Span(S). [Hint: this is the same as solving a certain system of equations]

We need to find weights x_1 and x_2 such that $x_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Equating coordinates gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & 1 \\ 2x_1 & -2x_2 & = & 2 \end{array}$$

Adding -1 times the first equation to the second gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & 1 \\ & -4x_2 & = & 1 \end{array}$$

We now divide the first equation by 2 and multiply the second equation by $-1/4$. Finally, we add -1 times the new second equation to the first equation, and this gives the echelon form

$$\begin{array}{rrcl} x_1 & & = & 3/4 \\ & x_2 & = & -1/4 \end{array}$$

We have thus shown that $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, proving that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in Span(S).

(b) Show that every vector (b_1, b_2) in $\mathbb{R}^2$ is in Span(S).

The calculation is very similar to that of part (a). Let $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, and look for weights x_1, x_2 satisfying $x_1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Equating coordinates gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & b_1 \\ 2x_1 & -2x_2 & = & b_2 \end{array}$$

Adding -1 times the first equation to the second gives

$$\begin{array}{rrcl} 2x_1 & +2x_2 & = & b_1 \\ & -4x_2 & = & b_2 - b_1 \end{array}$$

We now divide the first equation by 2 and multiply the second equation by $-1/4$. Finally, we add -1 times the new second equation to the first equation, and this gives the echelon form

$$\begin{array}{rrcl} x_1 & & = & (1/2)b_1 + 1/4(b_2 - b_1) \\ & x_2 & = & -1/4(b_2 - b_1) \end{array}$$

Thus the system of equations has a solution, proving that $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is in Span(S). Thus Span(S) $= \mathbb{R}^2$.

8. Let S be a non-empty subset of $\mathbb{R}^n$. Assume that each vector in Span(S) can be written in one and only one way as a linear combination of vectors in S. Show that S is linearly independent.

Suppse that we have $a_1\mathbf{x_1} + \cdots + a_n\mathbf{x_n} = \mathbf{0}$ with $\mathbf{x_i} \in S$. We know that $\mathbf{0}$ can also be written as the linear combination

$$0\mathbf{x_1} + \cdots + 0\mathbf{x_n}.$$

Since $\mathbf{0} \in \operatorname{Span}(S)$, our hypothesis gives us that there is only one way to write $\mathbf{0}$ as a linear combination of vectors in S. Therefore we must have $a_1 = 0, \ldots, a_n = 0$. Hence S is linearly independent.